Review: Linear Approximation - 11/7/16

1 Linear Approximation

Goal: Approximate the value of $f(b)$ for some b.

When using linear approximation to find the approximate value of $f(b)$:

1. Find an x value $x = a$ near b that is easy to evaluate.

2. Find the tangent line to the curve at $(a, f(a))$. That is, find $L(x) = f(a) + f'(a)(x - a)$. This is called the **linear approximation of f at a.**

3. Since $f(x) \approx L(x)$ for values of x near a, we find that $f(b) \approx L(b)$.

Example 1.0.1 *Let $f(x) = \sqrt{x+3}$. Use linear approximation to find $f(.98) = \sqrt{3.98}$. We first notice that 1 is very close to .98, so we will start by finding the tangent line at 1. We need a point and a slope. The point is $(1, \sqrt{4}) = (1, 2)$. The slope is $f'(1)$. We can use the power rule to see that $f'(x) = \frac{1}{2\sqrt{x+3}}$. Then $f'(1) = \frac{1}{4}$. Then the equation for the tangent line is $y - 2 = \frac{1}{4}(x - 1)$, so $y = \frac{1}{4}x + \frac{7}{4}$. When we're really close to $x = 1$, the tangent line is a good approximation for the curve, that is to say $f(x) \approx \frac{1}{4}x + \frac{7}{4}$ when x is near 1. Then $f(.98) \approx \frac{1}{4}(.98) + \frac{7}{4} = 1.995$. If we plug this into a calculator, we find that the actual value of $\sqrt{3.98} \approx 1.99499$.*

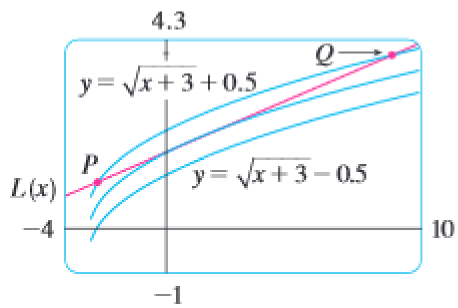
Example 1.0.2 *Let $f(x) = \sqrt{x+3}$, find $f(1.05) = \sqrt{4.05}$. We first notice that 1 is very close to 1.05, so we start by finding the tangent line at 1. Luckily, we've already done that: $y = \frac{1}{4}x + \frac{7}{4}$. Then $f(1.05) \approx \frac{1}{4}(1.05) + \frac{7}{4} = 2.0125$. If we plug this into a calculator, we find that the actual value of $\sqrt{4.05} \approx 2.012461$.*

Example 1.0.3 *Let $f(x) = e^x$, find $f(.1)$. We first notice that 0 is very close to .1, so we start by finding the tangent line at 0. Our point is $(0, 1)$, and our slope is $f'(0) = e^0 = 1$. Thus $y - 1 = 1(x - 0)$, so $y = x + 1$ is the equation for our tangent line. That means that for values of x close to 0, $f(x) \approx x + 1$. This is a good example of how linear approximation can be very useful - it is a lot easier to calculate what $x + 1$ is than e^x. Then $f(.1) \approx 1.1$. Below is a table of values of e^x according to our linear approximation and in reality. Notice how the farther from 0 we get, the less accurate our linear approximation becomes.*

x	***Linear Approximation of*** $f(x)$	***Actual Value of*** $f(x)$
0	1	1
.1	1.1	1.10517
.25	1.25	1.28402
.5	1.5	1.64872
1	2	2.71828

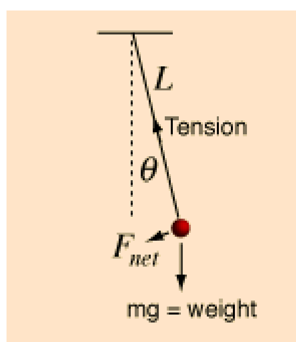
2 Error

As we just saw, approximations get less accurate the further away we are. The question is, how far away can we get? It depends on how accurate we want to be. Say we want to have an accuracy of ε. Then if we plot an ε window on either side of the function, everything inside of that band will be within ε correct. In the picture below, we are correct within .5 as long as the x value we choose is between P and Q.



3 Application: Motion of a Pendulum

In physics, if we look at a pendulum, we have that $a = -g\sin(\theta)$ where a is the acceleration, g is gravity, and θ is the angle from the center (see picture)



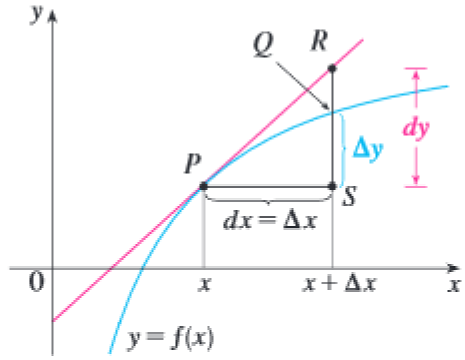
The $\sin(\theta)$ part of this makes the equation a little bit complicated. Let's use linear approximation to try to make it less complicated! Let $f(x) = \sin(x)$, and let's say we want to look at x near 0 (with our pendulum, we will often swing the pendulum from not too far away, so we can assume that θ is not too large). Then our point for the tangent line is $(0,0)$, and $f'(x) = \cos(x)$, so our slope is $f'(0) = \cos(0) = 1$. Thus $y - 0 = 1(x - 0)$, so $y = x$. Thus for small values of x, we have that $\sin(x) \approx x$. Going back to our pendulum, we see that for small values of θ, $a \approx -g\theta$. This is substantially easier to calculate.

4 Differentials

When talking about linearization, we often talk about **differentials** - changes to x and y. Recall that we can write $\frac{dy}{dx} = f'(x)$. If we interpret the dy and dx in terms of tiny changes in x and y, we can multiply both sides by dx to get that

$$dy = f'(x)dx.$$

Recall that we saw that Δy and Δx also mean changes in x and y. How do these compare with the differentials? We have that when $dx = \Delta x$, then dy is the change in height of the tangent line, whereas Δy is the change in height of the function.



Practice Problems

1. Let $f(x) = xe^{2x}$. Use linear approximation to find $f(.1)$.
2. Let $f(x) = \sqrt{x^2 - 3}$. Use linear approximation to find $f(2.2)$.
3. Let $f(x) = 5\cos(x)$. Use linear approximation to find $f(.05)$.

Solutions

1. Let's use $x = 0$ as our point for the tangent line. Then our point is $(0, f(0)) = (0, 0)$. Our slope is $f'(0)$. Since $f'(x) = e^{2x} + 2xe^{2x}$, then the slope is $f'(0) = 1$. Thus our tangent line is $y - 0 = 1(x - 0)$, so $y = x$. Thus $f(x) \approx x$ for x near zero, so $f(.1) \approx .1$. The actual value of this is $f(.1) \approx .122$.
2. Let's use $x = 2$ as our point for the tangent line. Then our point is $(2, 1)$. Our slope is $f'(2)$. Since $f'(x) = \frac{1}{2\sqrt{x^2-3}} \cdot 2x = \frac{x}{\sqrt{x^2-3}}$, then then slope is $f'(2) = 2$. Thus our tangent line is $y - 1 = 2(x - 2)$, so $y = 2x - 3$. Thus $f(2.2) \approx 2(2.2) - 3 = 1.4$. The actual value is $f(2.2) \approx 1.3565$.
3. Let's use $x = 0$ as our point for the tangent line. Then our point is $(0, 5)$. Our slope is $f'(0)$. Since $f'(x) = -5\sin(x)$, then our slope is $f'(0) = -5\sin(0) = 0$. Thus our tangent line is $y - 5 = 0(x - 0)$, so $y = 5$. Thus $f(.05) \approx 5$. The actual value is $f(.05) \approx 4.994$.